SC708: Hierarchical Linear Modeling
Instructor: Natasha Sarkisian
Class notes: Longitudinal Data with Complex Covariance Structures

So far, we took into account time as an independent variable, and we also accounted for the fact that multiple observations came from the same person (by having random intercepts). We also considered that the effects of time might vary from person to person (random slopes). But we did not take into account that data collected at a given time point can also be special in some way and different from data collected at another time point. Therefore, we need to assess the pattern of correlations/covariances of residuals across time points. Therefore, we will consider more complex covariance structures.

Examples of covariance structures for a dataset with six time points (t1-t6):

Independent = OLS regression (1 parameter for residual variance):

	t_1	t_2	t_3	t_4	t_5	t_6
t_1	—	0	0	0	0	0
t_2	0	—	0	0	0	0
t_3	0	0	—	0	0	0
t_4	0	0	0	—	0	0
t_5	0	0	0	0	—	0
t_6	0	0	0	0	0	—

Homogenous, random intercept only (2 parameters for residual variance, σ^2 and τ_{00}):

	t_1	t_2	t_3	t_4	t_5	t_6
t_1	—	ρ	ρ	ρ	ρ	ρ
t_2	ρ	—	ρ	ρ	ρ	ρ
t_3	ρ	ρ	—	ρ	ρ	ρ
t_4	ρ	ρ	ρ	—	ρ	ρ
t_5	ρ	ρ	ρ	ρ	—	ρ
t_6	ρ	ρ	ρ	ρ	ρ	—

First-order autoregressive (three parameters for residual variance, σ^2 , τ_{00} , and ρ):

	t_1	t_2	t_3	t_4	t_5	t_6
t_1	—	ρ^1	ρ^2	ρ^3	ρ^4	ρ^5
t_2	ρ^1	—	ρ^1	ρ^2	ρ^3	ρ^4
t_3	ρ^2	ρ^1	—	ρ^1	ρ^2	ρ^3
t_4	ρ^3	ρ^2	ρ^1	—	ρ^1	ρ^2
t_5	ρ^4	ρ^3	ρ^2	ρ^1	—	ρ^1
t_6	ρ^5	ρ^4	ρ^3	ρ^2	ρ^1	—

Unrestricted (six separate σ^2 parameters for residual variance and 15 covariances):

	t_1	t_2	t_3	t_4	t_5	t_6
t_1	—	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
t_2	ρ_1	—	ρ_6	ρ_7	ρ_8	ρ_9
t_3	ρ_2	ρ_6	—	ρ_{10}	ρ_{11}	ρ_{12}
t_4	ρ_3	ρ_7	ρ_{10}	—	ρ_{13}	ρ_{14}
t_5	ρ_4	ρ_8	ρ_{11}	ρ_{13}	—	ρ_{15}
t_6	ρ_5	ρ_9	ρ_{12}	ρ_{14}	ρ_{15}	—

In order to be able to model some of these more complex structures, we will need to create and use HMLM data files. To do that, we will use nys2.dta file on the course website – it's the same data we used in our previous example but slightly reorganized so that we can also learn some data management.

```
. use "C:\Users\sarkisin\nys2.dta", clear
```

The dataset is in wide format and we need to change it into long in order to be able to use for our analyses.

```
. reshape long attit expo, i(id) j(age)
(note: j = 14 15 16 17 18)

Data                               wide   ->   long
-----------------------------------------------------------------------------
Number of obs.                      241   ->    1205
Number of variables                  15   ->       8
j variable (5 values)                     ->   age
xij variables:
            attit14 attit15 ... attit18   ->   attit
               expo14 expo15 ... expo18   ->   expo
-----------------------------------------------------------------------------
```

Sometimes we might want to change the format into wide, however – for example, when using MICE to impute missing data in longitudinal datasets, you would want to use wide format to do the imputation, and afterwards, change it back into long.

```
. reshape wide attit expo, i(id) j(age)
(note: j = 14 15 16 17 18)

Data                               long   ->   wide
-----------------------------------------------------------------------------
Number of obs.                     1205   ->     241
Number of variables                   8   ->      15
j variable (5 values)               age   ->   (dropped)
xij variables:
                                  attit   ->   attit14 attit15 ... attit18
                                   expo   ->   expo14 expo15 ... expo18
-----------------------------------------------------------------------------
```

Let's go back to long format:

```
. reshape long attit expo, i(id) j(age)
(note: j = 14 15 16 17 18)
Data                               wide   ->   long
-----------------------------------------------------------------------------
Number of obs.                      241   ->    1205
Number of variables                  15   ->       8
j variable (5 values)                     ->   age
xij variables:
            attit14 attit15 ... attit18   ->   attit
               expo14 expo15 ... expo18   ->   expo
-----------------------------------------------------------------------------

. tab age
        age |      Freq.     Percent        Cum.
------------+-----------------------------------
         14 |        241       20.00       20.00
         15 |        241       20.00       40.00
         16 |        241       20.00       60.00
         17 |        241       20.00       80.00
         18 |        241       20.00      100.00
------------+-----------------------------------
      Total |      1,205      100.00

. gen age16=age-16
. gen age16s=age16^2
```

In order to model different covariance structures, we need to create indicator dummies for each time point.

```
. tab age, gen(ind)
        age |      Freq.     Percent        Cum.
------------+-----------------------------------
         14 |        241       20.00       20.00
         15 |        241       20.00       40.00
         16 |        241       20.00       60.00
         17 |        241       20.00       80.00
         18 |        241       20.00      100.00
------------+-----------------------------------
      Total |      1,205      100.00
```

We would be running into a missing data problem because we created blank observations when shifting from wide into long. Let's get rid of those lines that do not have level 1 data:

```
. drop if expo==. & attit==.
(139 observations deleted)
```

HLM does not like the new format of Stata so we will transfer the file into SPSS and import it.

```
. saveold "C:\Users\sarkisin\nys_ind.dta", replace
file C:\Users\sarkisin\nys_ind.dta saved
```

Now we will import it into HLM using HMLM format; note that you will have to specify indicator variables on level 1 – those are time point indicators we created.

```
                      LEVEL-1 DESCRIPTIVE STATISTICS

 VARIABLE NAME       N       MEAN         SD         MINIMUM      MAXIMUM
       AGE         1066      15.97        1.40        14.00        18.00
     ATTIT         1066       0.49        0.28         0.00         1.39
      EXPO         1066       0.56        0.31         0.00         1.61
     AGE16         1066      -0.03        1.40        -2.00         2.00
    AGE16S         1066       1.97        1.68         0.00         4.00
      IND1         1066       0.20        0.40         0.00         1.00
      IND2         1066       0.20        0.40         0.00         1.00
      IND3         1066       0.21        0.41         0.00         1.00
      IND4         1066       0.20        0.40         0.00         1.00
      IND5         1066       0.19        0.39         0.00         1.00


                      LEVEL-2 DESCRIPTIVE STATISTICS

 VARIABLE NAME       N       MEAN         SD         MINIMUM      MAXIMUM
    FEMALE          241       0.43        0.50         0.00         1.00
  MINORITY          241       0.23        0.42         0.00         1.00
    INCOME          241       4.09        2.35         1.00        10.00
    MINFEM          241       0.07        0.25         0.00         1.00


***UNRESTRICTED MODEL VS HOMOGENOUS MODEL WITHOUT RANDOM SLOPES***

  The outcome variable is    ATTIT

  The model specified for the fixed effects was:
 ----------------------------------------------------

   Level-1                  Level-2
   Coefficients             Predictors
```

```
 ----------------------   ---------------
         INTRCPT1, P0      INTRCPT2, B00
#     AGE16 slope, P1      INTRCPT2, B10
#    AGE16S slope, P2      INTRCPT2, B20

'#' - The residual parameter variance for this level-1 coefficient has been set
      to zero.

 OUTPUT FOR UNRESTRICTED MODEL

 Summary of the model specified (in equation format)
 ---------------------------------------------------

Level-1 Model

       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00
       P1 = B10
       P2 = B20

Var(E) = D

Iterations stopped due to small change in likelihood function

******* ITERATION 7 *******

 D
     IND1       0.08605      0.05148      0.04310      0.03083      0.03154
     IND2       0.05148      0.08446      0.05417      0.04144      0.03253
     IND3       0.04310      0.05417      0.08190      0.05504      0.04450
     IND4       0.03083      0.04144      0.05504      0.06953      0.04562
     IND5       0.03154      0.03253      0.04450      0.04562      0.07458

D (as correlations)
     IND1  1.000  0.604  0.513  0.399  0.394
     IND2  0.604  1.000  0.651  0.541  0.410
     IND3  0.513  0.651  1.000  0.729  0.569
     IND4  0.399  0.541  0.729  1.000  0.634
     IND5  0.394  0.410  0.569  0.634  1.000

 Standard Errors of D
     IND1       0.00822      0.00674      0.00636      0.00573      0.00604
     IND2       0.00674      0.00809      0.00664      0.00592      0.00597
     IND3       0.00636      0.00664      0.00768      0.00625      0.00613
     IND4       0.00573      0.00592      0.00625      0.00667      0.00585
     IND5       0.00604      0.00597      0.00613      0.00585      0.00735

The value of the likelihood function at iteration 7 = 8.389074E+001

 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                     Standard             Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00          0.511104   0.017166    29.774       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.032141   0.005279     6.088       240    0.000
 For   AGE16S slope, P2
```

```
   INTRCPT2, B20          -0.010502   0.003634    -2.890       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -167.78148
 Number of estimated parameters =   18

 OUTPUT FOR RANDOM EFFECTS MODEL WITH HOMOGENEOUS LEVEL-1 VARIANCE

 Summary of the model specified (in equation format)
 ---------------------------------------------------
Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00 + R0
       P1 = B10
       P2 = B20

Var(E) = Var(A*R + e) = D = A*Tau*A' + S
where S = sigma_squared*I

 A
     IND1       1.00000
     IND2       1.00000
     IND3       1.00000
     IND4       1.00000
     IND5       1.00000

Iterations stopped due to small change in likelihood function

******* ITERATION 4 *******

                    Parameter     Standard Error
                   ------------  ----------------
 sigma_squared =      0.03573         0.001758

 Tau
 INTRCPT1       0.04430

Tau (as correlations)
 INTRCPT1  1.000

 Standard Errors of Tau
 INTRCPT1       0.00484

 D
     IND1       0.08003      0.04430      0.04430      0.04430      0.04430
     IND2       0.04430      0.08003      0.04430      0.04430      0.04430
     IND3       0.04430      0.04430      0.08003      0.04430      0.04430
     IND4       0.04430      0.04430      0.04430      0.08003      0.04430
     IND5       0.04430      0.04430      0.04430      0.04430      0.08003

D (as correlations)
     IND1  1.000  0.554  0.554  0.554  0.554
     IND2  0.554  1.000  0.554  0.554  0.554
     IND3  0.554  0.554  1.000  0.554  0.554
     IND4  0.554  0.554  0.554  1.000  0.554
     IND5  0.554  0.554  0.554  0.554  1.000

The value of the likelihood function at iteration 4 = 4.104257E+001
```

```
 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                            Standard               Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00          0.513093   0.016359    31.364       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.032214   0.004234     7.608      1063    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20         -0.010343   0.003489    -2.964      1063    0.003
 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =    -82.08514
 Number of estimated parameters =    5

Summary of Model Fit
 -------------------------------------------------------------------
 Model                               Number of        Deviance
                                     Parameters
 -------------------------------------------------------------------
 1. Unrestricted                         18          -167.78148
 2. Homogeneous sigma_squared             5           -82.08514

 -------------------------------------------------------------------
 Model Comparison                Chi_square       df    P-value
 -------------------------------------------------------------------
 Model 1 vs Model 2                85.69633      13     0.000
```

Clearly, unrestricted model performs much better than the homogenous variable with random intercept. Also, when examining the pattern of variances/covariances and correlations in the unrestricted model, we see that variances are fairly stable over time and that correlations are stronger for time points that are closer to each other. Next, we will compare the unrestricted model to that with random slopes.

```
***UNRESTRICTED MODEL VS HOMOGENOUS MODEL WITH RANDOM SLOPES***

  The outcome variable is    ATTIT

  The model specified for the fixed effects was:
 ----------------------------------------------------
   Level-1                  Level-2
   Coefficients             Predictors
 ----------------------   ---------------
         INTRCPT1, P0      INTRCPT2, B00
      AGE16 slope, P1      INTRCPT2, B10
     AGE16S slope, P2      INTRCPT2, B20

 OUTPUT FOR UNRESTRICTED MODEL

 Summary of the model specified (in equation format)
 ---------------------------------------------------
Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00
       P1 = B10
       P2 = B20
```

```
Var(E) = D

Iterations stopped due to small change in likelihood function
******* ITERATION 7 *******

 D
     IND1       0.08605      0.05148      0.04310      0.03083      0.03154
     IND2       0.05148      0.08446      0.05417      0.04144      0.03253
     IND3       0.04310      0.05417      0.08190      0.05504      0.04450
     IND4       0.03083      0.04144      0.05504      0.06953      0.04562
     IND5       0.03154      0.03253      0.04450      0.04562      0.07458

D (as correlations)
     IND1  1.000  0.604  0.513  0.399  0.394
     IND2  0.604  1.000  0.651  0.541  0.410
     IND3  0.513  0.651  1.000  0.729  0.569
     IND4  0.399  0.541  0.729  1.000  0.634
     IND5  0.394  0.410  0.569  0.634  1.000

 Standard Errors of D
     IND1       0.00822      0.00674      0.00636      0.00573      0.00604
     IND2       0.00674      0.00809      0.00664      0.00592      0.00597
     IND3       0.00636      0.00664      0.00768      0.00625      0.00613
     IND4       0.00573      0.00592      0.00625      0.00667      0.00585
     IND5       0.00604      0.00597      0.00613      0.00585      0.00735

The value of the likelihood function at iteration 7 = 8.389074E+001

 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                         Standard              Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00          0.511104   0.017166    29.774       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.032141   0.005279     6.088       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20         -0.010502   0.003634    -2.890       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -167.78148
 Number of estimated parameters =   18

 OUTPUT FOR RANDOM EFFECTS MODEL WITH HOMOGENEOUS LEVEL-1 VARIANCE

 Summary of the model specified (in equation format)
 ---------------------------------------------------
Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00 + R0
       P1 = B10 + R1
       P2 = B20 + R2

Var(E) = Var(A*R + e) = D = A*Tau*A' + S
where S = sigma_squared*I
```

```
 A
     IND1       1.00000     -2.00000      4.00000
     IND2       1.00000     -1.00000      1.00000
     IND3       1.00000      0.00000      0.00000
     IND4       1.00000      1.00000      1.00000
     IND5       1.00000      2.00000      4.00000

Iterations stopped due to small change in likelihood function
******* ITERATION 5 *******

                    Parameter     Standard Error
                   ------------  ----------------
 sigma_squared =      0.02291         0.001618

 Tau
 INTRCPT1       0.05795     -0.00033     -0.00413
    AGE16      -0.00033      0.00366     -0.00033
   AGE16S      -0.00413     -0.00033      0.00117

Tau (as correlations)
 INTRCPT1  1.000 -0.023 -0.502
    AGE16 -0.023  1.000 -0.161
   AGE16S -0.502 -0.161  1.000

 Standard Errors of Tau
 INTRCPT1       0.00655      0.00141      0.00112
    AGE16       0.00141      0.00063      0.00029
   AGE16S       0.00112      0.00029      0.00031

 D
     IND1       0.08782      0.05229      0.04209      0.03432      0.02897
     IND2       0.05229      0.07876      0.05415      0.04720      0.03500
     IND3       0.04209      0.05415      0.08086      0.05350      0.04078
     IND4       0.03432      0.04720      0.05350      0.07611      0.04632
     IND5       0.02897      0.03500      0.04078      0.04632      0.07452

D (as correlations)
     IND1  1.000  0.629  0.499  0.420  0.358
     IND2  0.629  1.000  0.679  0.610  0.457
     IND3  0.499  0.679  1.000  0.682  0.525
     IND4  0.420  0.610  0.682  1.000  0.615
     IND5  0.358  0.457  0.525  0.615  1.000

The value of the likelihood function at iteration 5 = 7.620696E+001

 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                         Standard             Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00          0.514014   0.017270    29.764       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.031468   0.005320     5.915       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20         -0.010694   0.003643    -2.935       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -152.41391
 Number of estimated parameters =   10
```

```
Summary of Model Fit
 -------------------------------------------------------------------
 Model                               Number of        Deviance
                                     Parameters
 -------------------------------------------------------------------
 1. Unrestricted                         18          -167.78148
 2. Homogeneous sigma_squared            10          -152.41391

 -------------------------------------------------------------------
 Model Comparison                Chi_square       df    P-value
 -------------------------------------------------------------------
 Model 1 vs Model 2                 15.36757       8     0.052
```

The advantage of unrestricted model is no longer that clear – in fact, with .05 cutoff, the difference is not statistically significant. Also note, however, that this model is much closer than the previous one to the unrestricted model in terms of the number of parameters. Let's examine some additional models – we'll start with a model that allows for level 1 variance (sigma squared) to vary freely across time points.

```
****HETEROGENOUS MODEL WITH NO RANDOM SLOPES***

  The outcome variable is    ATTIT

  The model specified for the fixed effects was:
 ----------------------------------------------------
   Level-1                  Level-2
   Coefficients             Predictors
 ----------------------   ---------------
         INTRCPT1, P0      INTRCPT2, B00
#     AGE16 slope, P1      INTRCPT2, B10
#    AGE16S slope, P2      INTRCPT2, B20

'#' - The residual parameter variance for this level-1 coefficient has been set
      to zero.
 OUTPUT FOR UNRESTRICTED MODEL

 Summary of the model specified (in equation format)
 ---------------------------------------------------
Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00
       P1 = B10
       P2 = B20

Var(E) = D

Iterations stopped due to small change in likelihood function
******* ITERATION 7 *******

 D
     IND1       0.08605      0.05148      0.04310      0.03083      0.03154
     IND2       0.05148      0.08446      0.05417      0.04144      0.03253
     IND3       0.04310      0.05417      0.08190      0.05504      0.04450
     IND4       0.03083      0.04144      0.05504      0.06953      0.04562
     IND5       0.03154      0.03253      0.04450      0.04562      0.07458

D (as correlations)
     IND1  1.000  0.604  0.513  0.399  0.394
```

```
     IND2  0.604  1.000  0.651  0.541  0.410
     IND3  0.513  0.651  1.000  0.729  0.569
     IND4  0.399  0.541  0.729  1.000  0.634
     IND5  0.394  0.410  0.569  0.634  1.000

 Standard Errors of D
     IND1       0.00822      0.00674      0.00636      0.00573      0.00604
     IND2       0.00674      0.00809      0.00664      0.00592      0.00597
     IND3       0.00636      0.00664      0.00768      0.00625      0.00613
     IND4       0.00573      0.00592      0.00625      0.00667      0.00585
     IND5       0.00604      0.00597      0.00613      0.00585      0.00735

The value of the likelihood function at iteration 7 = 8.389074E+001
The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                       Standard             Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00          0.511104   0.017166    29.774       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.032141   0.005279     6.088       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20         -0.010502   0.003634    -2.890       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -167.78148
 Number of estimated parameters =   18

 OUTPUT FOR RANDOM EFFECTS MODEL WITH HOMOGENEOUS LEVEL-1 VARIANCE

 Summary of the model specified (in equation format)
 ---------------------------------------------------

Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00 + R0
       P1 = B10
       P2 = B20

Var(E) = Var(A*R + e) = D = A*Tau*A' + S
where S = sigma_squared*I

 A
     IND1       1.00000
     IND2       1.00000
     IND3       1.00000
     IND4       1.00000
     IND5       1.00000

Iterations stopped due to small change in likelihood function
****** ITERATION 4 *******
                    Parameter     Standard Error
                   ------------  ----------------
 sigma_squared =      0.03573         0.001758

 Tau
 INTRCPT1       0.04430
```

```
Tau (as correlations)
 INTRCPT1  1.000

 Standard Errors of Tau
 INTRCPT1       0.00484

 D
     IND1       0.08003      0.04430      0.04430      0.04430      0.04430
     IND2       0.04430      0.08003      0.04430      0.04430      0.04430
     IND3       0.04430      0.04430      0.08003      0.04430      0.04430
     IND4       0.04430      0.04430      0.04430      0.08003      0.04430
     IND5       0.04430      0.04430      0.04430      0.04430      0.08003

D (as correlations)
     IND1  1.000  0.554  0.554  0.554  0.554
     IND2  0.554  1.000  0.554  0.554  0.554
     IND3  0.554  0.554  1.000  0.554  0.554
     IND4  0.554  0.554  0.554  1.000  0.554
     IND5  0.554  0.554  0.554  0.554  1.000

The value of the likelihood function at iteration 4 = 4.104257E+001

 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                       Standard             Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00          0.513093   0.016359    31.364       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.032214   0.004234     7.608      1063    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20         -0.010343   0.003489    -2.964      1063    0.003

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =    -82.08514
 Number of estimated parameters =    5

 OUTPUT FOR RANDOM EFFECTS MODEL WITH HETEROGENEOUS LEVEL-1 VARIANCE

 Summary of the model specified (in equation format)
 ---------------------------------------------------

Level-1 Model
      Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
      Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
      P0 = B00 + R0
      P1 = B10
      P2 = B20

Var(E) = Var(A*R + e) = D = A*Tau*A' + S
where S = diag(sigma_squared(1),...,sigma_squared(5))

 A
     IND1       1.00000
     IND2       1.00000
     IND3       1.00000
     IND4       1.00000
```

```
     IND5       1.00000

Iterations stopped due to small change in likelihood function

******* ITERATION 8 *******
                     sigma_
                     squared      Standard Error
                   ------------  ----------------
     IND1            0.05411         0.005966
     IND2            0.03866         0.004507
     IND3            0.02437         0.003103
     IND4            0.02282         0.003027
     IND5            0.03875         0.004560
 Tau
 INTRCPT1       0.04637

Tau (as correlations)
 INTRCPT1  1.000

 Standard Errors of Tau
 INTRCPT1       0.00496

 D
     IND1       0.10048      0.04637      0.04637      0.04637      0.04637
     IND2       0.04637      0.08503      0.04637      0.04637      0.04637
     IND3       0.04637      0.04637      0.07074      0.04637      0.04637
     IND4       0.04637      0.04637      0.04637      0.06919      0.04637
     IND5       0.04637      0.04637      0.04637      0.04637      0.08512

D (as correlations)
     IND1  1.000  0.502  0.550  0.556  0.501
     IND2  0.502  1.000  0.598  0.605  0.545
     IND3  0.550  0.598  1.000  0.663  0.598
     IND4  0.556  0.605  0.663  1.000  0.604
     IND5  0.501  0.545  0.598  0.604  1.000

The value of the likelihood function at iteration 8 = 5.556423E+001

 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                         Standard             Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00          0.511249   0.016033    31.888       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.030756   0.004564     6.739      1063    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20         -0.009776   0.003531    -2.769      1063    0.006


 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -111.12847
 Number of estimated parameters =    9

Summary of Model Fit
 -------------------------------------------------------------------
 Model                                Number of        Deviance
                                      Parameters
 -------------------------------------------------------------------
 1. Unrestricted                          18          -167.78148
```

```
 2. Homogeneous sigma_squared             5           -82.08514
 3. Heterogeneous sigma_squared           9          -111.12847

 ------------------------------------------------------------------
 Model Comparison                 Chi_square       df    P-value
 ------------------------------------------------------------------
 Model 1 vs Model 2                  85.69633      13     0.000
 Model 1 vs Model 3                  56.65301       9     0.000
 Model 2 vs Model 3                  29.04332       4     0.000
```

Heterogenous model performs better than homogenous but worse than unrestricted. Let's try heterogenous with random slopes.

```
***HETEROGENOUS MODEL WITH RANDOM SLOPES***

  The outcome variable is    ATTIT

  The model specified for the fixed effects was:
 ----------------------------------------------------

   Level-1                  Level-2
   Coefficients             Predictors
 ----------------------   ---------------
        INTRCPT1, P0      INTRCPT2, B00
     AGE16 slope, P1      INTRCPT2, B10
    AGE16S slope, P2      INTRCPT2, B20

 OUTPUT FOR UNRESTRICTED MODEL

 Summary of the model specified (in equation format)
---------------------------------------------------
Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00
       P1 = B10
       P2 = B20

Var(E) = D

Iterations stopped due to small change in likelihood function
******* ITERATION 7 *******

 D
     IND1       0.08605      0.05148      0.04310      0.03083      0.03154
     IND2       0.05148      0.08446      0.05417      0.04144      0.03253
     IND3       0.04310      0.05417      0.08190      0.05504      0.04450
     IND4       0.03083      0.04144      0.05504      0.06953      0.04562
     IND5       0.03154      0.03253      0.04450      0.04562      0.07458

D (as correlations)
     IND1  1.000  0.604  0.513  0.399  0.394
     IND2  0.604  1.000  0.651  0.541  0.410
     IND3  0.513  0.651  1.000  0.729  0.569
     IND4  0.399  0.541  0.729  1.000  0.634
     IND5  0.394  0.410  0.569  0.634  1.000

 Standard Errors of D
     IND1       0.00822      0.00674      0.00636      0.00573      0.00604
     IND2       0.00674      0.00809      0.00664      0.00592      0.00597
     IND3       0.00636      0.00664      0.00768      0.00625      0.00613
```

```
     IND4       0.00573      0.00592      0.00625      0.00667      0.00585
     IND5       0.00604      0.00597      0.00613      0.00585      0.00735

The value of the likelihood function at iteration 7 = 8.389074E+001

 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                        Standard             Approx.
    Fixed Effect         Coefficient    Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00           0.511104   0.017166    29.774       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10           0.032141   0.005279     6.088       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20          -0.010502   0.003634    -2.890       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -167.78148
 Number of estimated parameters =   18

 OUTPUT FOR RANDOM EFFECTS MODEL WITH HOMOGENEOUS LEVEL-1 VARIANCE

 Summary of the model specified (in equation format)
 ---------------------------------------------------

Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00 + R0
       P1 = B10 + R1
       P2 = B20 + R2

Var(E) = Var(A*R + e) = D = A*Tau*A' + S
where S = sigma_squared*I

 A
     IND1       1.00000     -2.00000      4.00000
     IND2       1.00000     -1.00000      1.00000
     IND3       1.00000      0.00000      0.00000
     IND4       1.00000      1.00000      1.00000
     IND5       1.00000      2.00000      4.00000

Iterations stopped due to small change in likelihood function
******* ITERATION 5 *******

                    Parameter     Standard Error
                   ------------  ----------------
 sigma_squared =      0.02291         0.001618

 Tau
 INTRCPT1       0.05795     -0.00033     -0.00413
    AGE16      -0.00033      0.00366     -0.00033
   AGE16S      -0.00413     -0.00033      0.00117


Tau (as correlations)
 INTRCPT1  1.000 -0.023 -0.502
    AGE16 -0.023  1.000 -0.161
```

```
   AGE16S -0.502 -0.161  1.000

 Standard Errors of Tau
 INTRCPT1       0.00655      0.00141      0.00112
    AGE16       0.00141      0.00063      0.00029
   AGE16S       0.00112      0.00029      0.00031

 D
     IND1       0.08782      0.05229      0.04209      0.03432      0.02897
     IND2       0.05229      0.07876      0.05415      0.04720      0.03500
     IND3       0.04209      0.05415      0.08086      0.05350      0.04078
     IND4       0.03432      0.04720      0.05350      0.07611      0.04632
     IND5       0.02897      0.03500      0.04078      0.04632      0.07452
D (as correlations)
     IND1  1.000  0.629  0.499  0.420  0.358
     IND2  0.629  1.000  0.679  0.610  0.457
     IND3  0.499  0.679  1.000  0.682  0.525
     IND4  0.420  0.610  0.682  1.000  0.615
     IND5  0.358  0.457  0.525  0.615  1.000

The value of the likelihood function at iteration 5 = 7.620696E+001

 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                       Standard             Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00          0.514014   0.017270    29.764       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.031468   0.005320     5.915       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20         -0.010694   0.003643    -2.935       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -152.41391
 Number of estimated parameters =   10

 OUTPUT FOR RANDOM EFFECTS MODEL WITH HETEROGENEOUS LEVEL-1 VARIANCE

 Summary of the model specified (in equation format)
 ---------------------------------------------------

Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00 + R0
       P1 = B10 + R1
       P2 = B20 + R2

Var(E) = Var(A*R + e) = D = A*Tau*A' + S
where S = diag(sigma_squared(1),...,sigma_squared(5))

 A
     IND1       1.00000     -2.00000      4.00000
     IND2       1.00000     -1.00000      1.00000
     IND3       1.00000      0.00000      0.00000
     IND4       1.00000      1.00000      1.00000
     IND5       1.00000      2.00000      4.00000
```

```
Iterations stopped due to small change in likelihood function
******* ITERATION 6 *******

                     sigma_
                     squared      Standard Error
                   ------------  ----------------
     IND1            0.01371         0.008807
     IND2            0.03317         0.004204
     IND3            0.01853         0.003178
     IND4            0.01839         0.002911
     IND5            0.02192         0.007036

 Tau
 INTRCPT1       0.05891      0.00023     -0.00447
    AGE16       0.00023      0.00397     -0.00068
   AGE16S      -0.00447     -0.00068      0.00144

Tau (as correlations)
 INTRCPT1  1.000  0.015 -0.485
    AGE16  0.015  1.000 -0.286
   AGE16S -0.485 -0.286  1.000

 Standard Errors of Tau
 INTRCPT1       0.00658      0.00142      0.00113
    AGE16       0.00142      0.00077      0.00046
   AGE16S       0.00113      0.00046      0.00038

 D
     IND1       0.08583      0.05368      0.04057      0.03280      0.03036
     IND2       0.05368      0.08946      0.05421      0.04745      0.03600
     IND3       0.04057      0.05421      0.07744      0.05467      0.04150
     IND4       0.03280      0.04745      0.05467      0.07286      0.04686
     IND5       0.03036      0.03600      0.04150      0.04686      0.07401

D (as correlations)
     IND1  1.000  0.613  0.498  0.415  0.381
     IND2  0.613  1.000  0.651  0.588  0.442
     IND3  0.498  0.651  1.000  0.728  0.548
     IND4  0.415  0.588  0.728  1.000  0.638
     IND5  0.381  0.442  0.548  0.638  1.000

The value of the likelihood function at iteration 6 = 8.187945E+001

 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                          Standard             Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00          0.512145   0.017272    29.652       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.031974   0.005277     6.059       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20         -0.010570   0.003631    -2.911       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -163.75891
 Number of estimated parameters =   14

Summary of Model Fit
```

```
 -------------------------------------------------------------------
 Model                               Number of        Deviance
                                     Parameters
 -------------------------------------------------------------------
 1. Unrestricted                        18          -167.78148
 2. Homogeneous sigma_squared           10          -152.41391
 3. Heterogeneous sigma_squared         14          -163.75891

 -------------------------------------------------------------------
 Model Comparison                Chi_square       df    P-value
 -------------------------------------------------------------------
 Model 1 vs Model 2               15.36757       8     0.052
 Model 1 vs Model 3                4.02257       4     0.404
 Model 2 vs Model 3               11.34500       4     0.023
```

Heterogenous sigma squared is performing as well as unrestricted. But its number of parameters is also pretty high. Let's examine an autoregressive model.

```
***AUTOREGRESSIVE MODEL***

  The outcome variable is    ATTIT

  The model specified for the fixed effects was:
 ----------------------------------------------------

   Level-1                  Level-2
   Coefficients             Predictors
 ----------------------   ---------------
         INTRCPT1, P0      INTRCPT2, B00
      AGE16 slope, P1      INTRCPT2, B10
     AGE16S slope, P2      INTRCPT2, B20

 OUTPUT FOR UNRESTRICTED MODEL

 Summary of the model specified (in equation format)
 ---------------------------------------------------
Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00
       P1 = B10
       P2 = B20

Var(E) = D

Iterations stopped due to small change in likelihood function
******* ITERATION 7 *******
 D
     IND1       0.08605      0.05148      0.04310      0.03083      0.03154
     IND2       0.05148      0.08446      0.05417      0.04144      0.03253
     IND3       0.04310      0.05417      0.08190      0.05504      0.04450
     IND4       0.03083      0.04144      0.05504      0.06953      0.04562
     IND5       0.03154      0.03253      0.04450      0.04562      0.07458

D (as correlations)
     IND1  1.000  0.604  0.513  0.399  0.394
     IND2  0.604  1.000  0.651  0.541  0.410
     IND3  0.513  0.651  1.000  0.729  0.569
     IND4  0.399  0.541  0.729  1.000  0.634
     IND5  0.394  0.410  0.569  0.634  1.000
```

```
 Standard Errors of D
     IND1      0.00822      0.00674      0.00636      0.00573      0.00604
     IND2      0.00674      0.00809      0.00664      0.00592      0.00597
     IND3      0.00636      0.00664      0.00768      0.00625      0.00613
     IND4      0.00573      0.00592      0.00625      0.00667      0.00585
     IND5      0.00604      0.00597      0.00613      0.00585      0.00735

The value of the likelihood function at iteration 7 = 8.389074E+001

 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                       Standard             Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00           0.511104   0.017166    29.774       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10           0.032141   0.005279     6.088       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20          -0.010502   0.003634    -2.890       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -167.78148
 Number of estimated parameters =   18

 OUTPUT FOR RANDOM EFFECTS MODEL WITH HOMOGENEOUS LEVEL-1 VARIANCE

 Summary of the model specified (in equation format)
 ---------------------------------------------------

Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00 + R0
       P1 = B10 + R1
       P2 = B20 + R2

Var(E) = Var(A*R + e) = D = A*Tau*A' + S
where S = sigma_squared*I

 A
     IND1      1.00000     -2.00000      4.00000
     IND2      1.00000     -1.00000      1.00000
     IND3      1.00000      0.00000      0.00000
     IND4      1.00000      1.00000      1.00000
     IND5      1.00000      2.00000      4.00000

Iterations stopped due to small change in likelihood function

******* ITERATION 5 *******

                    Parameter     Standard Error
                   ------------  ----------------
 sigma_squared =      0.02291         0.001618

 Tau
 INTRCPT1       0.05795     -0.00033     -0.00413
    AGE16      -0.00033      0.00366     -0.00033
   AGE16S      -0.00413     -0.00033      0.00117
```

```
Tau (as correlations)
 INTRCPT1  1.000 -0.023 -0.502
    AGE16 -0.023  1.000 -0.161
   AGE16S -0.502 -0.161  1.000

 Standard Errors of Tau
 INTRCPT1       0.00655      0.00141      0.00112
    AGE16       0.00141      0.00063      0.00029
   AGE16S       0.00112      0.00029      0.00031

 D
     IND1       0.08782      0.05229      0.04209      0.03432      0.02897
     IND2       0.05229      0.07876      0.05415      0.04720      0.03500
     IND3       0.04209      0.05415      0.08086      0.05350      0.04078
     IND4       0.03432      0.04720      0.05350      0.07611      0.04632
     IND5       0.02897      0.03500      0.04078      0.04632      0.07452

D (as correlations)
     IND1  1.000  0.629  0.499  0.420  0.358
     IND2  0.629  1.000  0.679  0.610  0.457
     IND3  0.499  0.679  1.000  0.682  0.525
     IND4  0.420  0.610  0.682  1.000  0.615
     IND5  0.358  0.457  0.525  0.615  1.000

The value of the likelihood function at iteration 5 = 7.620696E+001
 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                       Standard             Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00           0.514014   0.017270    29.764       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10           0.031468   0.005320     5.915       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20          -0.010694   0.003643    -2.935       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -152.41391
 Number of estimated parameters =   10

 OUTPUT FOR RANDOM EFFECTS MODEL FIRST-ORDER AUTOREGRESSIVE MODEL
 FOR LEVEL-1 VARIANCE

 Summary of the model specified (in equation format)
 ---------------------------------------------------

Level-1 Model

      Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
      Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
      P0 = B00 + R0
      P1 = B10 + R1
      P2 = B20 + R2

Var(E) = Var(A*R + e) = D = A*Tau*A' + S
where S = {sigma_squared*rho**|t - t'|}
```

```
 A
     IND1       1.00000     -2.00000      4.00000
     IND2       1.00000     -1.00000      1.00000
     IND3       1.00000      0.00000      0.00000
     IND4       1.00000      1.00000      1.00000
     IND5       1.00000      2.00000      4.00000

Iterations stopped due to small change in likelihood function
******* ITERATION 6 *******

                    Parameter     Standard Error
                   ------------  ----------------
 rho           =      0.11901     ,    0.121199
 sigma_squared =      0.02670         0.004901

 Tau
 INTRCPT1       0.05420     -0.00031     -0.00342
    AGE16      -0.00031      0.00300     -0.00033
   AGE16S      -0.00342     -0.00033      0.00094


Tau (as correlations)
 INTRCPT1  1.000 -0.025 -0.480
    AGE16 -0.025  1.000 -0.194
   AGE16S -0.480 -0.194  1.000

 Standard Errors of Tau
 INTRCPT1       0.00789      0.00140      0.00136
    AGE16       0.00140      0.00095      0.00029
   AGE16S       0.00136      0.00029      0.00040

 D
     IND1       0.08703      0.05293      0.04152      0.03454      0.02984
     IND2       0.05293      0.07928      0.05427      0.04567      0.03522
     IND3       0.04152      0.05427      0.08090      0.05364      0.04026
     IND4       0.03454      0.04567      0.05364      0.07672      0.04713
     IND5       0.02984      0.03522      0.04026      0.04713      0.07409

D (as correlations)
     IND1  1.000  0.637  0.495  0.423  0.372
     IND2  0.637  1.000  0.678  0.586  0.460
     IND3  0.495  0.678  1.000  0.681  0.520
     IND4  0.423  0.586  0.681  1.000  0.625
     IND5  0.372  0.460  0.520  0.625  1.000

The value of the likelihood function at iteration 6 = 7.667277E+001

 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                         Standard             Approx.
    Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00         0.513903   0.017258    29.777       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10         0.031996   0.005303     6.033       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20        -0.010640   0.003649    -2.916       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -153.34553
```

```
 Number of estimated parameters =   11

Summary of Model Fit
 -------------------------------------------------------------------
 Model                               Number of         Deviance
                                     Parameters
 -------------------------------------------------------------------
 1. Unrestricted                         18          -167.78148
 2. Homogeneous sigma_squared            10          -152.41391
 3. First-order Autoregressive           11          -153.34553
 -------------------------------------------------------------------
 Model Comparison                Chi_square       df    P-value
 -------------------------------------------------------------------
 Model 1 vs Model 2                15.36757       8      0.052
 Model 1 vs Model 3                14.43595       7      0.043
 Model 2 vs Model 3                 0.93162       1      >.500
```

Autoregressive model is significantly different from the unrestricted, and it has more parameters than the homogenous model with random slopes, so we would probably not choose this model. (We could also try autoregressive without random slopes, but since even the one with random slopes did not perform well, it's not worth it.) Let's examine one more model, a log-linear model where we use level 1 predictors to model sigma squared (level 1 variance). This is similar to heterogenous variance models we examined in regular HLM.

```
****LOG-LINEAR MODEL***

  The outcome variable is    ATTIT

  The model specified for the fixed effects was:
 ----------------------------------------------------
   Level-1                   Level-2
   Coefficients              Predictors
 ----------------------   ---------------
         INTRCPT1, P0      INTRCPT2, B00
      AGE16 slope, P1      INTRCPT2, B10
     AGE16S slope, P2      INTRCPT2, B20

 OUTPUT FOR UNRESTRICTED MODEL

 Summary of the model specified (in equation format)
 ---------------------------------------------------
Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00
       P1 = B10
       P2 = B20

Var(E) = D

Iterations stopped due to small change in likelihood function

******* ITERATION 7 *******
 D
     IND1       0.08605      0.05148      0.04310      0.03083      0.03154
     IND2       0.05148      0.08446      0.05417      0.04144      0.03253
     IND3       0.04310      0.05417      0.08190      0.05504      0.04450
     IND4       0.03083      0.04144      0.05504      0.06953      0.04562
     IND5       0.03154      0.03253      0.04450      0.04562      0.07458
```

```
D (as correlations)
     IND1  1.000  0.604  0.513  0.399  0.394
     IND2  0.604  1.000  0.651  0.541  0.410
     IND3  0.513  0.651  1.000  0.729  0.569
     IND4  0.399  0.541  0.729  1.000  0.634
     IND5  0.394  0.410  0.569  0.634  1.000

 Standard Errors of D
     IND1       0.00822      0.00674      0.00636      0.00573      0.00604
     IND2       0.00674      0.00809      0.00664      0.00592      0.00597
     IND3       0.00636      0.00664      0.00768      0.00625      0.00613
     IND4       0.00573      0.00592      0.00625      0.00667      0.00585
     IND5       0.00604      0.00597      0.00613      0.00585      0.00735
The value of the likelihood function at iteration 7 = 8.389074E+001
 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                       Standard             Approx.
    Fixed Effect        Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00          0.511104   0.017166    29.774       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.032141   0.005279     6.088       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20         -0.010502   0.003634    -2.890       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -167.78148
 Number of estimated parameters =   18

 OUTPUT FOR RANDOM EFFECTS MODEL WITH HOMOGENEOUS LEVEL-1 VARIANCE

 Summary of the model specified (in equation format)
 ---------------------------------------------------
Level-1 Model
      Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
      Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
      P0 = B00 + R0
      P1 = B10 + R1
      P2 = B20 + R2

Var(E) = Var(A*R + e) = D = A*Tau*A' + S
where S = sigma_squared*I

 A
     IND1       1.00000     -2.00000      4.00000
     IND2       1.00000     -1.00000      1.00000
     IND3       1.00000      0.00000      0.00000
     IND4       1.00000      1.00000      1.00000
     IND5       1.00000      2.00000      4.00000

Iterations stopped due to small change in likelihood function
******* ITERATION 5 *******

                    Parameter     Standard Error
                   ------------  ----------------
 sigma_squared =      0.02291         0.001618
```

```
 Tau
 INTRCPT1       0.05795     -0.00033     -0.00413
    AGE16      -0.00033      0.00366     -0.00033
   AGE16S      -0.00413     -0.00033      0.00117

Tau (as correlations)
 INTRCPT1  1.000 -0.023 -0.502
    AGE16 -0.023  1.000 -0.161
   AGE16S -0.502 -0.161  1.000

 Standard Errors of Tau
 INTRCPT1       0.00655      0.00141      0.00112
    AGE16       0.00141      0.00063      0.00029
   AGE16S       0.00112      0.00029      0.00031
 D
     IND1       0.08782      0.05229      0.04209      0.03432      0.02897
     IND2       0.05229      0.07876      0.05415      0.04720      0.03500
     IND3       0.04209      0.05415      0.08086      0.05350      0.04078
     IND4       0.03432      0.04720      0.05350      0.07611      0.04632
     IND5       0.02897      0.03500      0.04078      0.04632      0.07452

D (as correlations)
     IND1  1.000  0.629  0.499  0.420  0.358
     IND2  0.629  1.000  0.679  0.610  0.457
     IND3  0.499  0.679  1.000  0.682  0.525
     IND4  0.420  0.610  0.682  1.000  0.615
     IND5  0.358  0.457  0.525  0.615  1.000

The value of the likelihood function at iteration 5 = 7.620696E+001

 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                       Standard             Approx.
   Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
 For       INTRCPT1, P0
    INTRCPT2, B00          0.514014   0.017270    29.764       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.031468   0.005320     5.915       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20         -0.010694   0.003643    -2.935       240    0.004

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -152.41391
 Number of estimated parameters =   10

 OUTPUT FOR RANDOM EFFECTS MODEL FOR LOG-LINEAR MODEL FOR LEVEL-1 VARIANCE

 Summary of the model specified (in equation format)
 ---------------------------------------------------
Level-1 Model
       Y = IND1*Y1* + IND2*Y2* + IND3*Y3* + IND4*Y4* + IND5*Y5*
       Y* = P0 + P1*(AGE16) + P2*(AGE16S) + e

Level-2 Model
       P0 = B00 + R0
       P1 = B10 + R1
       P2 = B20 + R2

Var(E) = Var(A*R + e) = D = A*Tau*A' + S
where S = diag(sigma_squared(1),...,sigma_squared(5)),
```

```
and log(sigma_squared(t)) = alpha0 + alpha1(AGE16) + alpha2(AGE16S)

 A
     IND1       1.00000     -2.00000      4.00000
     IND2       1.00000     -1.00000      1.00000
     IND3       1.00000      0.00000      0.00000
     IND4       1.00000      1.00000      1.00000
     IND5       1.00000      2.00000      4.00000

Iterations stopped due to small change in likelihood function
******* ITERATION 8 *******

                    Parameter     Standard Error
                   ------------  ----------------
 alpha0        =     -3.78640        0.098498
 alpha1        =     -0.13469        0.066674
 alpha2        =     -0.00472        0.066559

                     sigma_
                     squared
                   ------------
     IND1             0.02913
     IND2             0.02582
     IND3             0.02268
     IND4             0.01973
     IND5             0.01700

 Tau
 INTRCPT1       0.05777     -0.00018     -0.00416
    AGE16      -0.00018      0.00364     -0.00008
   AGE16S      -0.00416     -0.00008      0.00121

Tau (as correlations)
 INTRCPT1  1.000 -0.012 -0.497
    AGE16 -0.012  1.000 -0.036
   AGE16S -0.497 -0.036  1.000

 Standard Errors of Tau
 INTRCPT1       0.00653      0.00140      0.00111
    AGE16       0.00140      0.00072      0.00033
   AGE16S       0.00111      0.00033      0.00034

 D
     IND1       0.08954      0.05011      0.04150      0.03458      0.02934
     IND2       0.05011      0.08065      0.05379      0.04702      0.03452
     IND3       0.04150      0.05379      0.08044      0.05343      0.04078
     IND4       0.03458      0.04702      0.05343      0.07352      0.04811
     IND5       0.02934      0.03452      0.04078      0.04811      0.07352

D (as correlations)
     IND1  1.000  0.590  0.489  0.426  0.362
     IND2  0.590  1.000  0.668  0.611  0.448
     IND3  0.489  0.668  1.000  0.695  0.530
     IND4  0.426  0.611  0.695  1.000  0.654
     IND5  0.362  0.448  0.530  0.654  1.000

The value of the likelihood function at iteration 8 = 7.845170E+001
 The outcome variable is    ATTIT

 Final estimation of fixed effects:
                                           Standard                Approx.
   Fixed Effect         Coefficient   Error      T-ratio   d.f.     P-value
 ----------------------------------------------------------------------------
```

```
 For       INTRCPT1, P0
    INTRCPT2, B00          0.512555   0.017227    29.753       240    0.000
 For    AGE16 slope, P1
    INTRCPT2, B10          0.031346   0.005322     5.890       240    0.000
 For   AGE16S slope, P2
    INTRCPT2, B20         -0.010028   0.003650    -2.747       240    0.006

 Statistics for current covariance components model
 --------------------------------------------------
 Deviance =   -156.90340
 Number of estimated parameters =   12

Summary of Model Fit
 -------------------------------------------------------------------
 Model                               Number of        Deviance
                                     Parameters
 -------------------------------------------------------------------
 1. Unrestricted                         18          -167.78148
 2. Homogeneous sigma_squared            10          -152.41391
 3. Log-linear                           12          -156.90340
 -------------------------------------------------------------------
 Model Comparison                Chi_square       df    P-value
 -------------------------------------------------------------------
 Model 1 vs Model 2               15.36757       8     0.052
 Model 1 vs Model 3               10.87807       6     0.091
 Model 2 vs Model 3                4.48949       2     0.104
```

This model is not significantly different from either the unrestricted model or the homogenous model with random slopes. Overall, however, we would select the homogenous model with random slopes because it is more parsimonious. For that model, we would be able to go back to regular HLM. Note, that fixed effects and significance tests did not change much from model to model – in many cases, they are pretty robust to variance specifications. Still, it is a good idea to examine these – if the variance patterns are very pronounced and very far from homogenous, we could be seriously misspecifying the model. At a minimum, it is useful to inspect the covariance matrix and the results of the unrestricted model (unless the number of time points is very high because this model can become too complex). Also note that here we did that for the simplest model (with only age and age squared variables in the model), but it could be useful to check the variance structures for such a simple model but then also doublecheck it once more after building the final model.